Module 4

Single-phase AC Circuits

Lesson
14
Solution of Current in R-L-C Series Circuits

In the last lesson, two points were described:

1. How to represent a sinusoidal (ac) quantity, i.e. voltage/current by a phasor?
2. How to perform elementary mathematical operations, like addition/ subtraction and multiplication/division, of two or more phasors, represented as complex quantity?

Some examples are also described there. In this lesson, the solution of the steady state currents in simple circuits, consisting of resistance R, inductance L and/or capacitance C connected in series, fed from single phase ac supply, is presented. Initially, only one of the elements R / L / C, is connected, and the current, both in magnitude and phase, is computed. Then, the computation of total reactance and impedance, and the current, in the circuit consisting of two components, R & L / C only in series, is discussed. The process of drawing complete phasor diagram with current(s) and voltage drops in the different components is described. Lastly, the computation of total power and also power consumed in the components, along with the concept of power factor, is explained.

Keywords: Series circuits, reactance, impedance, phase angle, power, power factor.

After going through this lesson, the students will be able to answer the following questions;

1. How to compute the total reactance and impedance of the R-L-C series circuit, fed from single phase ac supply of known frequency?
2. How to compute the current and also voltage drops in the components, both in magnitude and phase, of the circuit?
3. How to draw the complete phasor diagram, showing the current and voltage drops?
4. How to compute the total power and also power consumed in the components, along with power factor?

Solution of Steady State Current in Circuits Fed from Single-phase AC Supply

Elementary Circuits

1. Purely resistive circuit (R only)

The instantaneous value of the current though the circuit (Fig. 14.1a) is given by,

$$i = \frac{v}{R} = \frac{V_m}{R}\sin \omega t = I_m \sin \omega t$$

where,

I_m and V_m are the maximum values of current and voltage respectively.

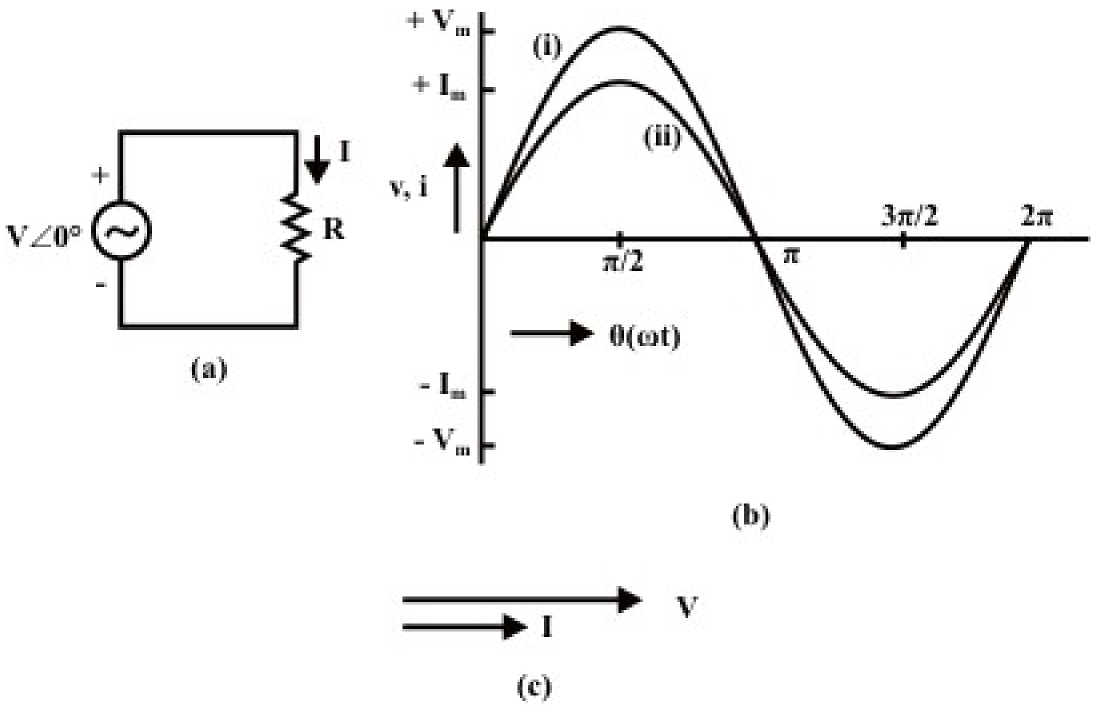

Fig. 14.1: Circuit with Resistance (R)
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

The rms value of current is given by

$$\bar{I} = \frac{I_m}{\sqrt{2}} = \frac{V_m/\sqrt{2}}{R} = \frac{\bar{V}}{R}$$

In phasor notation,

$$\bar{V} = V\angle 0° = V(1 + j\,0) = V + j\,0$$

$$\bar{I} = I\angle 0° = I(1 + j\,0) = I + j\,0$$

The impedance or resistance of the circuit is obtained as,

$$\frac{\bar{V}}{\bar{I}} = \frac{V\angle 0°}{I\angle 0°} = Z\angle 0° = R + j\,0$$

Please note that the voltage and the current are in phase ($\phi = 0°$), which can be observed from phasor diagram (Fig. 14.1b) with two (voltage and current) phasors, and also from the two waveforms (Fig. 14.1c).

In ac circuit, the term, Impedance is defined as voltage/current, as is the resistance in dc circuit, following Ohm's law. The impedance, Z is a complex quantity. It consists of real part as resistance R, and imaginary part as reactance X, which is zero, as there is no inductance/capacitance. All the components are taken as constant, having linear V-I characteristics. In the three cases being considered, including this one, the power consumed and also power factor in the circuits, are not taken up now, but will be described later in this lesson.

2. Purely inductive circuit (L only)

For the circuit (Fig. 14.2a), the current i, is obtained by the procedure described here.

As $v = L\dfrac{di}{dt} = V_m \sin\,\omega t = \sqrt{2}\,V \sin\,\omega t$,

$$di = \frac{\sqrt{2}\,V}{L}\sin(\omega t)\;dt$$

Integrating,

$$i = -\frac{\sqrt{2}\,V}{\omega L}\cos\,\omega t = \frac{\sqrt{2}\,V}{\omega L}\sin\,(\omega t - 90°) = I_m \sin\,(\omega t - 90°) = \sqrt{2}\,I \sin\,(\omega t - 90°)$$

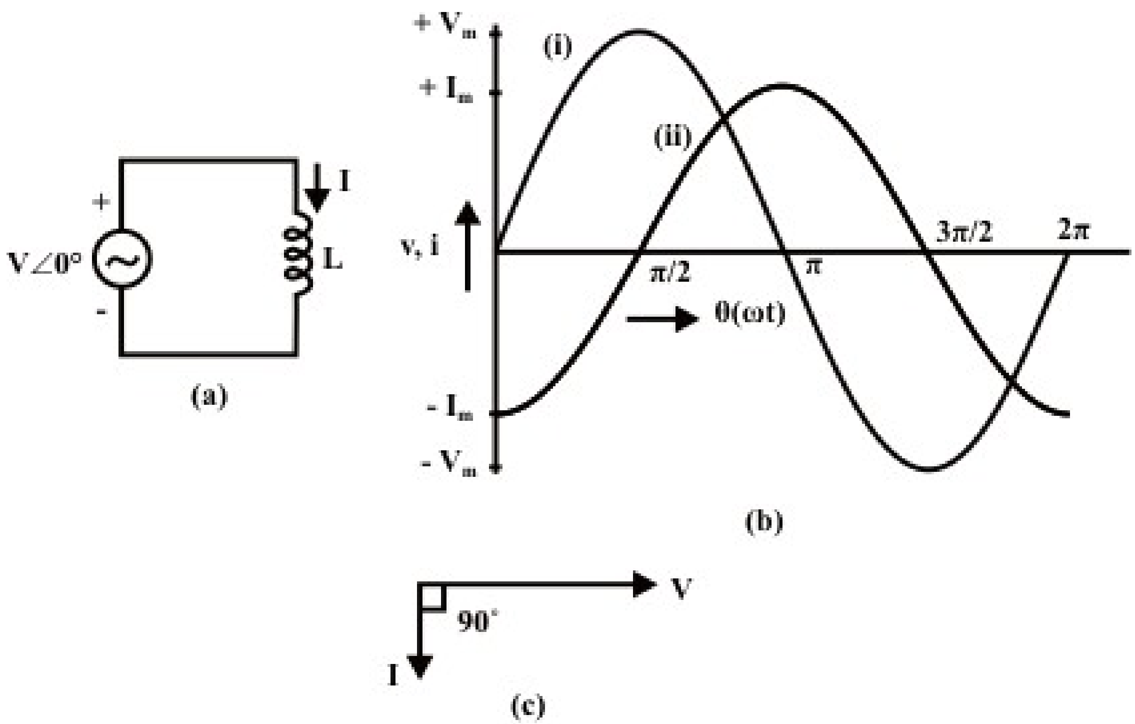

Fig. 14.2: Circuit with Inductance (L)
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

It may be mentioned here that the current i, is the steady state solution, neglecting the constant of integration. The rms value, I is

$$\bar{I} = \frac{\bar{V}}{\omega L} = I\angle -90°$$

$$\bar{V} = V\angle 0° = V + j\,0 \quad ; \qquad \bar{I} = I\angle -90° = 0 - j\,I$$

The impedance of the circuit is

$$Z\angle\phi = \frac{\bar{V}}{\bar{I}} = \frac{V\angle 0^\circ}{I\angle -90^\circ} = \frac{V}{-\,j\,I} = j\,\omega\,L = 0 + j\,X_L = X_L\,\angle 90^\circ = \omega\,L\angle 90^\circ$$

where, the inductive reactance is $X_L = \omega\,L = 2\,\pi\,f\,L$.

Note that the current lags the voltage by $\phi = +90^\circ$. This can be observed both from phasor diagram (Fig. 14.2b), and waveforms (Fig. 14.2c). As the circuit has no resistance, but only inductive reactance $X_L = \omega\,L$ (positive, as per convention), the impedance Z is only in the y-axis (imaginary).

3. Purely capacitive circuit (C only)

The current i, in the circuit (Fig. 14.3a), is,

$$i = C\frac{dv}{dt}$$

Substituting $v = \sqrt{2}\,V\sin\omega\,t = V_m\sin\omega\,t$, i is

$$i = C\frac{d}{dt}\left(\sqrt{2}\,V\sin\omega\,t\right) = \sqrt{2}\,\omega\,C\,V\cos\omega\,t = \sqrt{2}\,\omega\,C\,V\sin(\omega\,t + 90^\circ) = \sqrt{2}\,I\sin(\omega\,t + 90^\circ)$$

$$= I_m\sin(\omega\,t + 90^\circ)$$

The rms value, I is

$$\bar{I} = \omega\,C\,\bar{V} = \frac{\bar{V}}{1/(\omega\,C)} = I\,\angle 90^\circ$$

$$\bar{V} = V\,\angle 0^\circ = V + j\,0 \quad ; \qquad \bar{I} = I\,\angle 90^\circ = 0 + j\,I$$

The impedance of the circuit is

$$Z\angle\phi = \frac{\bar{V}}{\bar{I}} = \frac{V\angle 0^\circ}{I\angle 90^\circ} = \frac{V}{j\,I} = \frac{1}{j\,\omega\,C} = -\frac{j}{\omega\,C} = 0 - j\,X_C = X_C\,\angle -90^\circ = \frac{1}{\omega\,C}\angle 90^\circ$$

where, the capacitive reactance is $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\,\pi\,f\,C}$.

Note that the current leads the voltage by $\phi = 90^\circ$ (this value is negative, i.e. $\phi = -90^\circ$), as per convention being followed here. This can be observed both from phasor diagram (Fig. 14.3b), and waveforms (Fig. 14.3c). As the circuit has no resistance, but only capacitive reactance, $X_c = 1/(\omega\,C)$ (negative, as per convention), the impedance Z is only in the y-axis (imaginary).

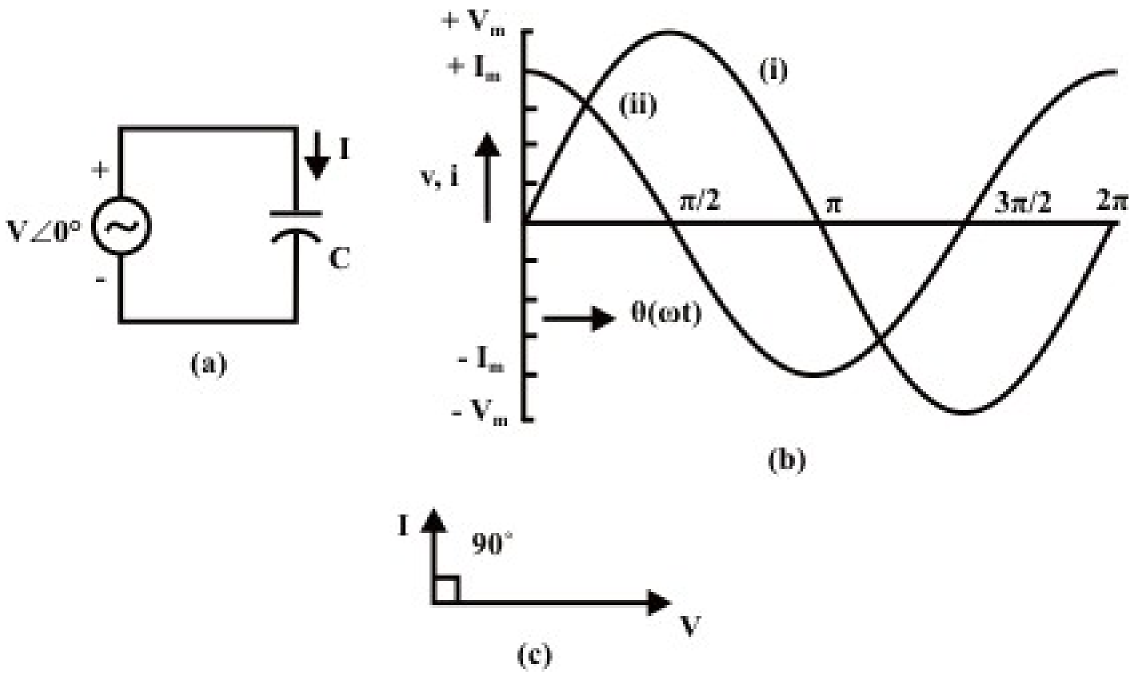

Fig. 14.3: Circuit with Capacitance (C)
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

Series Circuits

1. Inductive circuit (R and L in series)

The voltage balance equation for the R-L series circuit (Fig. 14.4a) is,

$$v = R\,i + L\frac{di}{dt}$$

where, $v = \sqrt{2}\,V\sin\omega t = V_m \sin\omega t = \sqrt{2}\,V\sin\theta$, θ being ωt.

The current, i (in steady state) can be found as

$$i = \sqrt{2}\,I\sin(\omega t - \phi) = I_m \sin(\omega t - \phi) = \sqrt{2}\,I\sin(\theta - \phi)$$

The current, $i(t)$ in steady state is sinusoidal in nature (neglecting transients of the form shown in the earlier module on dc transients). This can also be observed, if one sees the expression of the current, $i = I_m \sin(\omega t)$ for purely resistive case (with R only), and $i = I_m \sin(\omega t - 90°)$ for purely inductive case (with L only).

Alternatively, if the expression for i is substituted in the voltage equation, the equation as given here is obtained.

$$\sqrt{2}\,V\sin\omega t = R\cdot\sqrt{2}\,I\sin(\omega t - \phi) + \omega L\cdot\sqrt{2}\,I\cos(\omega t - \phi)$$

If, first, the trigonometric forms in the RHS side is expanded in terms of $\sin\omega t$ and $\cos\omega t$, and then equating the terms of $\sin\omega t$ and $\cos\omega t$ from two (LHS & RHS) sides, the two equations as given here are obtained.

$V = (R\cdot\cos\phi + \omega L\cdot\sin\phi)\cdot I$, and

$0=(-R\cdot\sin\phi+\omega L\cdot\cos\phi)$

From these equations, the magnitude and phase angle of the current, I are derived.

From the second one, $\tan\phi=(\omega L/R)$

So, phase angle, $\phi=\tan^{-1}(\omega L/R)$

Two relations, $\cos\phi=(R/Z)$, and $\sin\phi=(\omega L/Z)$, are derived, with the term (impedance), $Z=\sqrt{R^2+(\omega L)^2}$

If these two expressions are substituted in the first one, it can be shown that the magnitude of the current is $I=V/Z$, with both V and Z in magnitude only.

The steps required to find the rms value of the current I, using complex form of impedance, are given here.

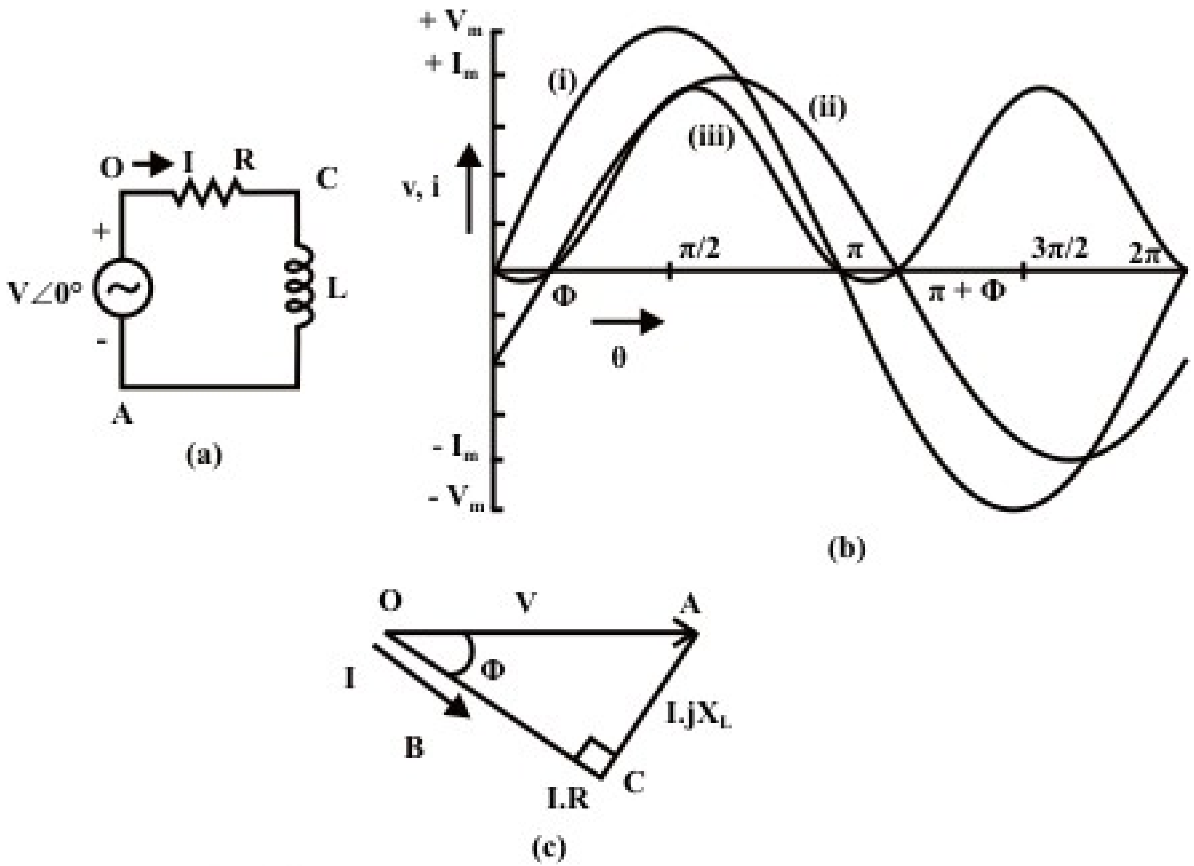

Fig. 14.4: Circuit with Resistance (R) and Inductance (L) in series.
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current (iii) Power
(c) Phasor diagram

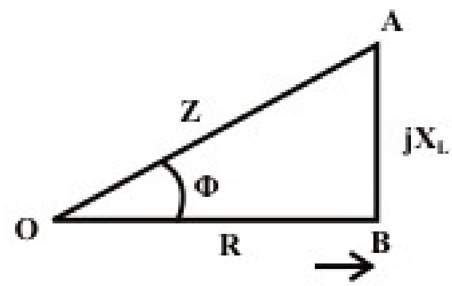

Fig. 14.5: The complex form of the impedance (R-L series circuit)

The impedance (Fig. 14.5) of the inductive (R-L) circuit is,

$$Z\angle\phi = R + j\,X_L = R + j\,\omega\,L$$

where,

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + (\omega\,L)^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}\left(\frac{\omega\,L}{R}\right)$$

$$\bar{I}\;\angle-\phi = \frac{\bar{V}\angle 0°}{Z\,\angle\phi} = \frac{V + j\,0}{R + j\,X_L} = \frac{V + j\,0}{R + j\,\omega\,L}$$

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_L^2}} = \frac{V}{\sqrt{R^2 + (\omega\,L)^2}}$$

Note that the current lags the voltage by the angle ϕ, value as given above. In this case, the voltage phasor has been taken as reference phase, with the current phasor lagging the voltage phasor by the angle, ϕ. But normally, in the case of the series circuit, the current phasor is taken as reference phase, with the voltage phasor leading the current phasor by ϕ. This can be observed both from phasor diagram (Fig. 14.4b), and waveforms (Fig. 14.4c). The inductive reactance X_L is positive. In the phasor diagram, as one move from voltage phasor to current phasor, one has to go in the clockwise direction, which means that phase angle, ϕ is taken as positive, though both phasors are assumed to move in anticlockwise direction as shown in the previous lesson.

The complete phasor diagram is shown in Fig. 14.4b, with the voltage drops across the two components and input (supply) voltage (OA), and also current (OB). The voltage phasor is taken as reference. It may be observed that

$$V_{OC}\,(=I\,R) + V_{CA}\,[=I\,(j\,X_L)] = V_{OA}\,(=I\,Z),$$

using the Kirchoff's second law relating to the voltage in a closed loop. The phasor diagram can also be drawn with the current phasor as reference, as will be shown in the next lesson.

Power consumed and Power factor

From the waveform of instantaneous power ($W = v \cdot i$) also shown in Fig. 14.4c for the above circuit, the average power is,

$$W = \frac{1}{\pi}\int_0^{\pi} v \cdot i \, d\theta = \frac{1}{\pi}\int_0^{\pi} \sqrt{2}V\sin\theta\sqrt{2}\,I\sin(\theta-\phi)\,d\theta = \frac{1}{\pi}\int_0^{\pi} V\,I\left[\cos\phi - \cos\,(2\theta-\phi)\right]d\theta$$

$$= \frac{1}{\pi}\left[V\,I\cos\phi\,\theta\Big|_0^{\pi} - \frac{V\,I}{2}\sin\,(2\theta-\phi)\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[V\,I\cos\phi(\pi-0) - \frac{V\,I}{2}\left[\sin(2\pi-\phi)+\sin\phi\right]\right] = V\,I\cos\phi$$

Note that power is only consumed in resistance, R only, but not in the inductance, L. So, $W = I^2 R$.

Power factor $= \dfrac{average\ power}{apparent\ power} = \dfrac{V\,I\cos\phi}{V\,I} = \cos\phi = \dfrac{R}{Z} = \dfrac{R}{\sqrt{R^2 + (\omega L)^2}}$

The power factor in this circuit is less than 1 (one), as $0° \le \phi \le 90°$, ϕ being positive as given above.

For the resistive (R) circuit, the power factor is 1 (one), as $\phi = 0°$, and the average power is $V\,I$.

For the circuits with only inductance, L or capacitance, C as described earlier, the power factor is 0 (zero), as $\phi = \pm 90°$. For inductance, the phase angle, or the angle of the impedance, $\phi = +90°$ (lagging), and for capacitance, $\phi = -90°$ (leading). It may be noted that in both cases, the average power is zero (0), which means that no power is consumed in the elements, L and C.

The complex power, Volt-Amperes (VA) and reactive power will be discussed after the next section.

2. Capacitive circuit (R and C in series)

This part is discussed in brief. The voltage balance equation for the R-C series circuit (Fig. 14.6a) is,

$$v = R\,i + \frac{1}{C}\int i\,dt = \sqrt{2}V\sin\omega t$$

The current is

$$i = \sqrt{2}\,I\sin(\omega t + \phi)$$

The reasons for the above choice of the current, i, and the steps needed for the derivation of the above expression, have been described in detail, in the case of the earlier example of inductive (R-L) circuit. The same set of steps has to be followed to derive the current, i in this case.

Alternatively, the steps required to find the rms value of the current I, using complex form of impedance, are given here.

The impedance of the capacitive (R-C) circuit is,

$$Z\angle -\phi = R - j\,X_C = R - j\,\frac{1}{\omega C}$$

where,

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \quad \text{and}$$

$$\phi = \tan^{-1}\left(-\frac{X_C}{R}\right) = \tan^{-1}\left(-\frac{1}{\omega C R}\right) = -\tan^{-1}\left(\frac{1}{\omega C R}\right)$$

$$\bar{I} \angle\phi = \frac{\bar{V}\angle 0^\circ}{Z\angle -\phi} = \frac{V + j\,0}{R - j\,X_C} = \frac{V + j\,0}{R - j\,(1/\omega C)}$$

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_C^2}} = \frac{V}{\sqrt{R^2 + (1/\omega C)^2}}$$

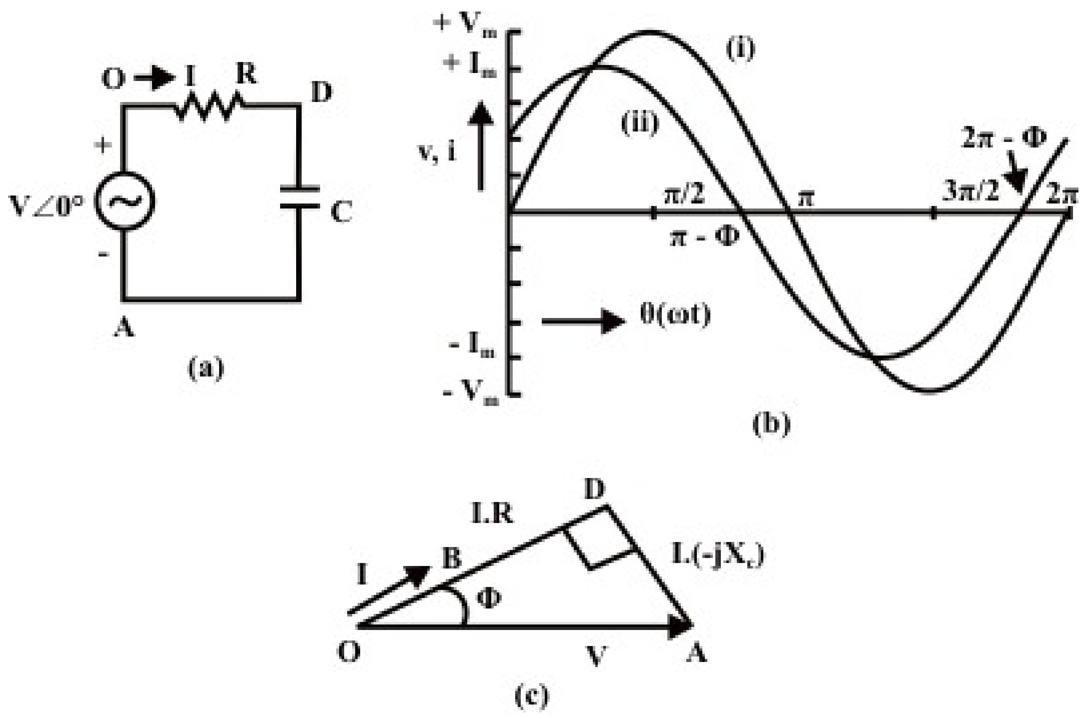

Fig. 14.6: Circuit with Resistance (R) and Capacitance (C) in series.
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

Note that the current leads the voltage by the angle ϕ, value as given above. In this case, the voltage phasor has been taken as reference phase, with the current phasor leading the voltage phasor by the angle, ϕ. But normally, in the case of the series circuit, the current phasor is taken as reference phase, with the voltage phasor lagging the current phasor by ϕ. This can be observed both from phasor diagram (Fig. 14.6b), and waveforms (Fig. 14.6c). The capacitive reactance X_C is negative. In the phasor diagram, as one move from voltage phasor to current phasor, one has to go in the anticlockwise direction, which means that phase angle, ϕ is taken as negative. This is in contrast to the case as described earlier. The complete phasor diagram is shown in Fig. 14.6b, with the voltage drops across the two components and input (supply) voltage, and also current. The voltage phasor is taken as reference.

The power factor in this circuit is less than 1 (one), with ϕ being same as given above. The expression for the average power is $P = V\,I\cos\phi$, which can be obtained by the method shown above. The power is only consumed in the resistance, R, but not in the capacitance, C. One example is included after the next section.

Complex Power, Volt-Amperes (VA) and Reactive Power

The complex power is the product of the voltage and complex conjugate of the current, both in phasor form. For the inductive circuit, described earlier, the voltage ($V\angle 0°$) is taken as reference and the current ($I\angle -\phi = I\cos\phi - j\,I\sin\phi$) is lagging the voltage by an angle, ϕ. The complex power is

$$\bar{S} = \bar{V}\cdot I^* = V\angle 0°\cdot I\angle\phi = (V\ I)\angle\phi = V\ I\cos\ \phi + jV\ I\sin\ \phi = P + j\,Q$$

The Volt-Amperes (S), a scalar quantity, is the product of the magnitudes the voltage and the current. So, $S = V\cdot I = \sqrt{P^2 + Q^2}$. It is expressed in VA.

The active power (W) is

$P = \mathrm{Re}\ (\bar{S}) = \mathrm{Re}\ (\bar{V}\cdot I^*) = V\,I\cos\,\phi$, as derived earlier.

The reactive power (VAr) is given by $Q = \mathrm{Im}\ (\bar{S}) = \mathrm{Im}\ (\bar{V}\cdot I^*) = V\,I\sin\,\phi$.

As the phase angle, ϕ is taken as positive in inductive circuits, the reactive power is positive. The real part, ($I\cos\phi$) is in phase with the voltage V, whereas the imaginary part, $I\sin\phi$ is in quadrature ($-90°$) with the voltage V. But in capacitive circuits, the current ($I\angle\phi$) leads the voltage by an angle ϕ, which is taken as negative. So, it can be stated that the reactive power is negative here, which can easily be derived

Example 14.1

A voltage of 120 V at 50 Hz is applied to a resistance, R in series with a capacitance, C (Fig. 14.7a). The current drawn is 2 A, and the power loss in the resistance is 100 W. Calculate the resistance and the capacitance.

Solution

V = 120 V $\quad$ I = 2 A $\quad$ P = 100 W $\quad$ f = 50 Hz

$$R = P / I^2 = 100 / 2^2 = 25\ \Omega$$

$$Z = \sqrt{R^2 + X_C^2} = V / I = 120 / 2 = 60\ \Omega$$

$$X_c = 1/(2\pi\ f\ C) = \sqrt{Z^2 - R^2} = \sqrt{(60)^2 - (25)^2} = 54.54\,\Omega$$

$$C = \frac{1}{2\pi\ f\ X_C} = \frac{1}{2\pi \cdot 50.0 \times 54.54} = 58.36 \cdot 10^{-6} = 58.36\ \mu F$$

The power factor is, $\cos\phi = R / Z = 25 / 60 = 0.417\ (lead)$

The phase angle is $\phi = \cos^{-1}(0.417) = 65.38°$

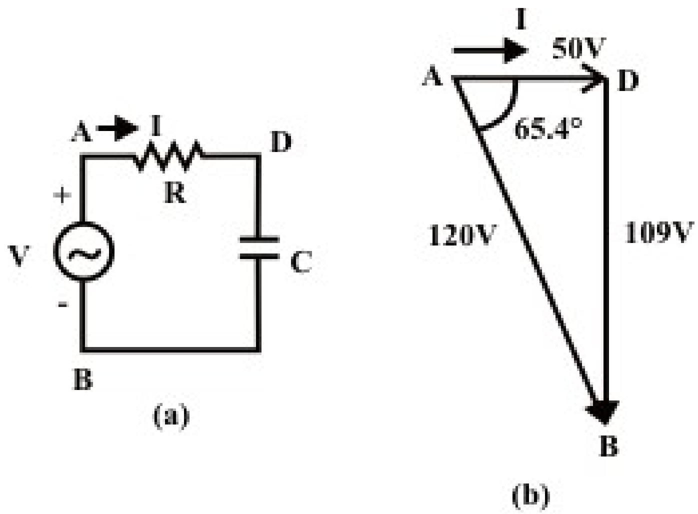

Fig. 14.7: (a) Circuit diagram
(b) Phasor diagram

The phasor diagram, with the current as reference, is shown in Fig. 14.7b. The examples, with lossy inductance coil (r in series with L), will be described in the next lesson. The series circuit with all elements, R. L & C, along with parallel circuits, will be taken up in the next lesson.

Problems

14.1 Calculate the power factor in the following cases for the circuit with the elements, as given, fed from a single phase ac supply.

(i) With resistance, R only, but no L and C

(a) 1.0 (Φ=0°) (b) 0.0 lagging (Φ=+90°)

(c) 0.0 leading (Φ=-90°) (d) None of the above

(ii) with only pure/lossless inductance, L, but no R and C

(a) 1.0 (Φ=0°) (b) 0.0 lagging (Φ=+90°)

(c) 0.0 leading (Φ=-90°) (d) None of the above

(iii) with only pure capacitance, C, but no R and L.

(a) 1.0 (Φ=0°) (b) 0.0 lagging (Φ=+90°)

(c) 0.0 leading (Φ=-90°) (d) None of the above

14.2 Calculate the current and power factor (lagging / leading) in the following cases for the circuits having impedances as given, fed from an ac supply of 200 V. Also draw the phasor diagram in all cases.

(i) $Z = (15+j20)\ \Omega$

(ii) $Z = (14-j14)\ \Omega$

(iii) $Z = R + j\,(X_L - X_C)$, where $R = 10\ \Omega$, $X_L = 20\ \Omega$, and $X_C = 10\ \Omega$.

14.3 A 200 V, 50 Hz supply is connected to a resistance (R) of 20 Ω in series with an iron cored choke coil (r in series with L). The readings of the voltmeters across the resistance and across the coil are 120 V and 150 V respectively. Find the loss in the coil. Also find the total power factor. Draw the phasor diagram.

14.4 A circuit, with a resistance, R and a lossless inductance in series, is connected across an ac supply (V) of known frequency (f). A capacitance, C is now connected in series with R-L, with V and f being constant. Justify the following statement with reasons.

The current in the circuit normally increases with the introduction of C.

Under what condition, the current may also decrease. Explain the condition with reasons.

List of Figures